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# Existence and Uniqueness Theory of the Vlasov Equation

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# Abstract

The purpose of this report is to communicate recent results on the existence and uniqueness of solutions to the Vlasov equation. Both the Vlasov-Poisson system and the Vlasov-Maxwell systems are considered. Chapter 1 is devoted to the Vlasov-Poisson system. A theorem is stated which gives conditions for the solvability of the system for general once differentiable initial data having compact support. As an example of the application of the general theorem the case where the system has spherical symmetry is considered. A global-in-time existence and uniqueness theorem is proved for this case. Chapter 2 deals with the Vlasov-Maxwell system. Here a local-in-time existence and uniqueness theorem is proved for a general class of initial data having compact support.

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# Table of Contents

| | | Page |
|---|---|---|
| Abstract | | i |
| Chapter 1 | The Vlasov-Poisson System | 1 |
| Chapter 2 | The Vlasov-Maxwell System | 20 |

# Chapter 1: The Vlasov-Poisson system

## I. Introduction

The Vlasov-Poisson system of equations describes the motion of a collection of particles in the presence of its own electrostatic (or gravitational) field. The system under consideration is

$$\text{a)}\ \frac{\partial f}{\partial t} + v\cdot\nabla_x f + \nabla_x \phi\cdot\nabla_v f = 0$$

$$(1.1) \qquad f(x,v,0) = g(x,v)$$

$$\text{b)}\ \Delta\phi = -4\pi^2 \int (f-b)dv$$

$(x,v)$ - a point in 6-D phase space

$f(x,v,t)$ - phase space distribution function

$b(x,v)$ - fixed background distribution

We assume $g,b \;\; C^1_0(R_6)$, i.e., continuously differentiable functions with bounded support. The support of a function is the region of phase space outside of which the function is zero. A function with bounded support goes to zero outside a bounded region of phase space. In this paper a general theorem is stated which gives conditions for the solvability of system (1.1). The proof of the theorem is given in [14]. The bibliography at the end of this chapter contains a set of references for the classical existence theory of (1.1). An application of the general theorem is made to the case where the initial data and hence the solution have spherical symmetry. The result is a global-in-time existence and uniqueness theorem for the system with spherical symmetry.

## II. General theorem

The following notation is used

$R_6$ - six dimensional phase space

$C_0^1(R_6)$ - the space of continuous functions with continuous bounded derivatives on $R_6$ and which have bounded support in $R_6$.

$C^1(R_6x[0,T])$ - The space of continuous functions with continuous bounded derivatives on $R_6x[0,T]$, T-time.

The term "solution" refers to a function that is of class $C^1$, satisfies (1.1), and which is integrable over phase space in such a way that appropriate conservation laws (like mass, momentum, energy) can hold.

Theorem: Let g,b be functions in $C_0^1(R_6)$. The solution to (1.1a,b) exists and is unique as an element of $C^1(R_6x[0,T])$ if and only if the system admits an a priori bound on the support of $C^1$ solutions for t in the interval [0,T].

Proof: see [14].

This theorem gives a criterion to determine whether the solution to (1.1a,b) exists and is unique for a particular initial configuration. To use the theorem let us make the assumption that a $C^1$ solution to the problem exists on the interval of time [0,T]. The theorem tells us that a $C^1$ solution to the problem necessarily has bounded support in phase space. We then try to compute a priori what the bound on the support of such a solution must be as a function of time. If it is possible to compute such an a priori bound for a solution, then the theorem tells us that the $C^1$ solution to the system in fact exists, is unique, and of course has the bound on support already computed. The important bound to compute is on velocity.

For initial data with bounded support the statement that the support of the solution remains bounded is equivalent to the statement that the electric field remains bounded. Therefore, an alternative way of stating the general theorem is as follows: a $C^1$ distribution function that is a solution to (1.1) cannot give rise to an unbounded electric field in a finite time. To prove existence and uniqueness of the solution to (1.1) of class $C^1(R_6 \times [0,T])$ it is sufficient to show "a priori" that the electric field remains bounded for $t \varepsilon [0,T]$.

## III. The spherically symmetric Vlasov-Poisson system

An existence and uniqueness theorem for the spherically symmetric case was first proved by Batt in [2]. The proof in [2] is valid for the gravitational case alone where the spherically symmetric field always points inward along a radius. In [13] a different proof is given which makes no use of an inward directed field and is thus valid for both the gravitational and electrostatic case. The proof in [13] builds, however, on some of the development in [2]. In this paper we give an independent development of the result in [13].

Let

$$r = (x_1^2 + x_2^2 + x_3^2)^{1/2}$$

$$u = (v_1^2 + v_2^2 + v_3^2)^{1/2}$$

$$\xi = x_1 v_1 + x_2 v_2 + x_3 v_3$$

Let the functions g,b in (1.1a,b) be of class $C_0^1(R_6)$ and have the form

$$(3.1) \qquad g(x,v) = \bar{G}(r,u,\xi) \ , \ b(x,v) = \bar{H}(r,u,\xi)$$

Initial and background distributions of this form are termed spherically symmetric, and solutions to (1.1) with data of this form reduce to functions in the variables r,u,ξ, and t and are termed spherically symmetric. Spherically symmetric solutions are characterized by the fact that the field remains a function only of radius and time. Also if (x(t), v(t)) is a trajectory of (1.1) then

$$x(t) \times v(t) = c \ .$$

These facts will become evident as we proceed.

Let

$$\eta = (x_1v_1 + x_2v_2 + x_3v_3)/_{ru} = \xi/_{ru} \ .$$

For points $(x,v) \varepsilon R_6$ such that $|\eta| \neq 1$ (i.e., the velocity vector does not point along the radius) spherically symmetric solutions to the Vlasov equation can be written in terms of the variables

$$r = (x_1^2 + x_2^2 + x_3^2)^{1/2}$$

$$u = (v_1^2 + v_2^2 + v_3^2)^{1/2}$$

$$\alpha = \cos^{-1}((x_1v_1 + x_2v_2 + x_3v_3)/ru)$$

Here α is the angle between the position vector and the velocity vector and takes on values between 0 and $\pi$. In terms of these variables the Vlasov-Poisson system reduces to the form

$$\text{a)}\quad \frac{\partial F}{\partial t} + \cos\alpha\, u\,\frac{\partial F}{\partial r} - \cos\alpha\,\Theta(t,r)\,\frac{\partial F}{\partial u}$$

$$+ (\Theta/u - u/r)\sin\alpha\,\frac{\partial F}{\partial \alpha} = 0$$

$$(3.2)\qquad F(0) = G$$

$$\text{b)}\quad \Theta(t,r) = -\frac{4\pi}{r^2}\int_0^r h(t,\bar{r})\,\bar{r}^2\,d\bar{r}$$

$$h(t,r) = 2\pi\int_0^\infty\int_0^\pi (F-H)\sin\alpha\,d\alpha\,u^2\,du$$

where

H(r,u,$\alpha$) is a fixed background distribution

h(t,r) the charge density,

$-\Theta$ the radial electric field

The characteristic equations for (3.2) are

$$\dot{r} = u\cos\alpha$$

$$(3.3)\qquad \dot{u} = -\Theta(t,r)\cos\alpha$$

$$\dot{\alpha} = (\Theta/u - u/r)\sin\alpha$$

An integral of (3.3) is

$$(3.4)\qquad r(t)u(t)\sin(\alpha(t)) = r(0)u(0)\sin(\alpha(0)) = c\ .$$

Thus if a trajectory of (3.2) (solution to (3.3)) has its initial point in the region $\eta \neq 1$ then the trajectory remains in the region $\eta \neq 1$.

That is $\eta \neq 1$ implies $r(0)u(0)\sin(\alpha(0)) = c \neq 0$. From (3.4) it follows that $r(t)u(t)\sin(\alpha(t)) \neq 0$ for all $t$ and hence the trajectory remains in the region $\eta \neq 1$ for all $t$. In order to prove the result of global existence and uniqueness for the spherically symmetric Vlasov-Poisson system we analyze trajectories in the region $\eta \neq 1$ and compute bounds on the trajectories in this region. By continuity results for the region $\eta \neq 1$ are extended to the lines $\eta = 1$. By this means an a priori bound on velocities in the system is computed.

## IV. Bounds on the field

Let the initial data G to (3.2a,b) have the bound

$$0 \leq G \leq D$$

and G and H have bounds

$$8\pi^2 \int_0^\infty \int_0^\infty \int_0^\pi G \sin\alpha \, d\alpha \, u^2 du \, r^2 dr \leq M/2 ,$$

$$8\pi^2 \int_0^\infty \int_0^\infty \int_0^\pi H \sin\alpha \, d\alpha \, u^2 du \, r^2 dr \leq M/2$$

Hence the solution to (3.2) has bounds

$$0 \leq F \leq D \quad , \; 8\pi^2 \int_0^\infty \int_0^\infty \int_0^\pi F \sin\alpha \, d\alpha \, u^2 du \, r^2 dr \leq M/2$$

and also

$$8\pi^2 \int_0^\infty \int_0^\infty \int_0^\pi |F - H| \sin\alpha \, d\alpha \, u^2 du \, r^2 dr \leq M$$

Let us assume a constant $\bar{A}$ such that

$$|h| \leq 2\pi \int |F - H| \sin\alpha \, d\alpha \, u^2 du \leq \bar{A}$$

The bounds D and M on F are known a priori from the initial data. The bound $\bar{A}$ gets computed as part of the proof.

Bounds for the field $\Theta$ are

$$|\Theta| \leq \frac{4\pi}{r^2} \int_0^r |h| \, \bar{r}^2 \, d\bar{r} \leq \frac{4\pi}{r^2} \bar{A} \, r^3/3$$

$$\leq Ar, \quad A = 4/3 \, \pi \, \bar{A}$$

$$|\Theta| \leq 1/_{r}2 \, (4\pi \int_0^r |h| \, \bar{r}^2 \, d\bar{r})$$

$$\leq 1/_{r}2 \, (4\pi \int_0^\infty |h| \bar{r}^2 \, d\bar{r}) \leq M/_{r}2$$

A continuous function bounding $\Theta$ is

$$g(r) = Ar \qquad 0 < r \leq (M/A)^{1/3} = \ell$$

$$g(r) = M/_{r}2 \qquad r \geq \ell$$

The continuous anti-derivative of g is

$$G(r) = 1/2 \; A \; r^2 - 3/2 \; A^{1/3} \; M^{2/3} \qquad r \leq \ell$$

$$G(r) = -M/r \qquad r \geq \ell$$

The bound on $\Theta$ is

$$|\Theta| \leq G'(r)$$

## V. Existence and uniqueness theorem for the spherically symmetric case

Theorem: Let the initial and background distributions g and b to (1.1a,b) be of class $C_0^1(R_6)$ and be of the form (3.1) (i.e., spherically symmetric). For such spherically symmetric data the solution to (1.1a,b) of class $C^1(R_6 \times [0,T])$ exists and is unique for all T.

Proof:

We state without proof that for initial and background data of form (3.1) the solutions to (1.1a,b) satisfy a system of the form (3.2a,b) in the variables $r,u,\alpha$, and $t$ in the region $\eta \neq 1$. As has been shown trajectories that originate in the region $\eta \neq 1$ remain in this region. An a priori bound on the support of the solution to (1.1a,b) is obtained by analyzing the trajectories (solutions to (3.3)) in the region $\eta \neq 1$. The results are extended by continuity to the lines $\eta = 1$.

Let us assume the existence of a solution F to (3.2a,b) on the interval of time $[0,T]$ which has bounded support in the variables $r,u$. We wish to compute a priori what this bound must be. Let

$\Omega(t) = \text{supp}\, F(t)$ - support of F in $r,u,\alpha$ space for each t.

$p = (r,u,\alpha)$ - a point in $r,u,\alpha$ space.

$E(p) = u$ - the projection of p on the u axis.

$$\bar{\Omega}(T) = \cup\Omega(t),\ t \in [0,T]$$

$$P(T) = \sup_{p \in \bar{\Omega}(T)} E(p)$$ - a bound on the support in velocity space up to time T.

Let the constant $\bar{A}(T)$ satisfy

$$|h| \leq 2\pi \int |F - H| \sin\alpha \, d\alpha \, u^2 du \leq \bar{A}(T) \text{ for } t \in [0,T]$$

and set

$$A = A(T) = 4/3\pi \; \bar{A}(T) .$$

We are assuming

$$8\pi^2 \int_0^\infty \int_0^\infty \int_0^\pi |F - H| \sin\alpha \, d\alpha \, u^2 du \; r^2 dr \leq M$$

Let

$$G(r) = 1/2 \, Ar^2 - 3/2 \, A^{1/3} M^{2/3} \qquad r \leq \ell = (M/A)^{1/3}$$

$$G(r) = -M/r \qquad r > \ell$$

The bound on the field $\Theta$ for $t \in [0,T]$ is

$$|\Theta| \leq G'(r)$$

For any two points $r_1, r_2$ the function $G(r)$ satisfies

$$(5.1) \qquad |G(r_1) - G(r_2)| \leq 3/2 \, M^{2/3} A^{1/3}$$

Let $p(t) = (r(t), u(t), \alpha(t))$ be a trajectory for $t \in [0,T]$ such that $p(0)$ is in the region $\eta \neq 1$. The proof now proceeds through a sequence of steps to get the estimate

$$u^2(t) \leqslant u^2(0) + 6M^{2/3}\,A^{1/3}$$

a) If $\dot{r} > 0$ or $\dot{r} \leqslant 0$ on $[t_1,t_2] \subset [0,T]$

then

$$|u^2(t_2) - u^2(t_1)| \leqslant 2\,|G(r(t_2)) - G(r(t_1))|$$

as follows:

$$\frac{d}{dt}(1/2\,u^2) = u\,\dot{u} = -\,\Theta\,(t,r)\,u\cos\alpha$$

$$= -\Theta\,\dot{r}$$

Hence

$$|u^2(t_2) - u^2(t_1)| \leqslant 2\int_{t_1}^{t_2} |\Theta\dot{r}|dt$$

However

if $\dot{r} > 0$ $\quad |\Theta\dot{r}| \leqslant \frac{d}{dt}G(r(t))$

if $\dot{r} \leqslant 0$ $\quad |\Theta\dot{r}| \leqslant -\frac{d}{dt}G(r(t))$

Thus

$$|u^2(t_2) - u^2(t_1)| \leq 2\ |G(r(t_2)) - G(r(t_1))|$$

b) At a local maximum of $r(t)$

$$u^2(t) \leq M^{2/3}\ A^{1/3}$$

as follows:

At a local maximum

$$\dot{r} = u \cos\alpha = 0 \rightarrow \alpha = \pi/2$$

$$\sin(\alpha) = 1$$

$$\ddot{r} = \dot{u} \cos\alpha - u \sin\alpha\ \dot{\alpha} = - u\ \dot{\alpha} \leq 0$$

$$\rightarrow \dot{\alpha} \geq 0$$

Hence from

$\dot{\alpha} = [\Theta/u - u/r] \sin\alpha$ we have

$$\Theta/u - u/r \geq 0 \quad \rightarrow \quad u^2 \leq \Theta r$$

Thus $\Theta \geq 0$ i.e. the field $-\Theta \leq 0$. From the bounds on $|\Theta|$

$$u^2 \leq Ar^2 \qquad r \leq \ell$$

$$u^2 \leqslant M/r \qquad r \geqslant \ell$$

The maximum occurs at $r = \ell$ and

$$u^2(t) \leqslant M^{2/3} A^{1/3}$$

c) At a local minimum of $r(t)$

$$u^2(t) \leqslant u^2(0) + 3M^{2/3} A^{1/3}$$

as follows: Let $t > 0$ be a local minimum and suppose $\ddot{r}(t) = 0$. From (3.3) we have that

$$\ddot{r} = \dot{u} \cos\alpha - u \sin(\alpha)\dot{\alpha}$$

$$= - \phi \cos^2\alpha - u \sin\alpha \; [\phi/u - u/r] \sin\alpha$$

$$= - \phi \cos^2\alpha - \phi \sin^2\alpha + \frac{u^2}{r} \sin^2\alpha$$

$$= - \phi + \frac{u^2}{r} \sin^2\alpha$$

At a minimum $\alpha = \pi/2$ and assuming $\ddot{r} = 0$ then

$$u^2 = \phi r$$

Thus from the bounds on $\phi$

$$u^2 \leq Ar^2 \qquad r \leq \ell$$

$$u^2 \leq M/r \qquad r \geq \ell$$

and letting $r = \ell$ it follows that

$$u^2(t) \leq M^{2/3} A^{1/3}$$

The estimate therefore holds if $\ddot{r}(t) = 0$.

Assume that $\ddot{r}(t) \neq 0$. Then there exists $t_0 < t$ such that either $t_0 = 0$ or $t_0$ is a local maximum and $\dot{r} \leq 0$ on $[t_0,t]$. If $t_0 = 0$ then from (5.1) and part a)

$$u^2(t) \leq u^2(0) + 3M^{2/3} A^{1/3}$$

If $t_0$ is a maximum we distinguish two cases:

i) $r(t),r(t_0) \leq \ell$ or $r(t),r(t_0) \geq \ell$ in which

$$|G(r(t)) - G(r(t_0))| \leq M^{2/3} A^{1/3}$$

and from parts a,b and (5.1)

$$u^2(t) \leqslant u^2(t_0) + 2M^{2/3} A^{1/3} \leqslant 3M^{2/3} A^{1/3}$$

ii) $r(t) \leqslant \ell$, $r(t_0) \geqslant \ell$ in which from (5.1) and parts a,b

$$u^2(t) \leqslant u^2(t_0) + 3M^{2/3} A^{1/3} \leqslant 4M^{2/3} A^{1/3}$$

We further refine this estimate.

$$(5.2) \qquad u^2(t) \leqslant \quad u^2(t_0) + 2|G(r(t)) - G(r(t_0))|$$

$$\leqslant u^2(t_0) + 3M^{2/3} A^{1/3} - \frac{2M}{r(t_0)} - A\, r(t)^2$$

Since $r(t_0)$, $r(t)$ are extrema then

$\alpha(t_0) = \alpha(t) = \pi/2$ and from (3.4)

$u(t_0)\, r(t_0) = u(t) r(t)$

$r(t_0) = r(t) u(t)/u(t_0)$

Substituting in (5.2) and minimizing the negative term over $r(t)$ we get

$$u^2(t) \leqslant u^2(t_0) + 3M^{2/3} A^{1/3} - 3M^{2/3} A^{1/3} \left(\frac{u(t_0)^2}{u(t)^2}\right)^{1/3}$$

But it is shown above that

$$u^2(t) < 4M^{2/3}\ A^{1/3}, \text{ thus}$$

$$(5.3) \qquad u^2(t) < u^2(t_0) + 3M^{2/3}\ A^{1/3} - 3/4^{1/3}\ M^{2/3}\ A^{1/3}\ \left(\frac{u(t_0)^2}{M^{2/3}A^{1/3}}\right)^{1/3}$$

It is easily seen that for $u^2(t_0)$ in the interval $[0, M^{2/3}A^{1/3}]$ (part b) the right side of (5.3) is maximized at $u^2(t_0) = 0$. Thus

$$u^2(t) < 3M^{2/3}\ A^{1/3}$$

This estimate is obtained in [2] through a different derivation.

d) For any $t > 0$

$$u^2(t) < u^2(0) + 6M^{2/3}\ A^{1/3}$$

as follows: If $\ddot{r}(t) = 0$ then as before

$$u^2(t) < M^{2/3}\ A^{1/3}$$

and the estimate holds. Assume therefore that $\ddot{r}(t) \neq 0$. If $t > 0$ there exists $t_0 < t$ such that either $t_0 = 0$ or $t_0$ is a local extremum of $r(t)$ and $\dot{r}$ does not change sign on $[t_0,t]$. Thus from (5.1) and parts a-c

$$5.4) \qquad u^2(t) < u^2(t_0) + 2|G(r(t)) - G(r(t_0))|$$

$$< u^2(0) + 6M^{2/3}\ A^{1/3}$$

Estimate (5.4) is now used to get an a priori bound on the support of F in velocity space. In terms of the bound on velocity

$$P = P(T) \quad \text{for} \quad t \in [0,T]$$

the charge density h has the bound

$$|h| < 2\pi \int_0^\infty \int_0^\pi |F - H| \sin\alpha \, d\alpha \, u^2 du$$

$$< \frac{4\pi}{3} D \, P^3(T) + B < \frac{4\pi}{3} (D + B) P^3(T) < \frac{4\pi}{3} C P^3(T)$$

(assuming $P(T) > 1$). Here $D = \sup|F|$ ,

$$2\pi \int_0^\infty \int_0^\pi |H| \sin\alpha \, d\alpha \, u^2 du < B \; , \; C = D + B$$

Therefore let

$$\bar{A}(T) = \frac{4\pi}{3} C \, P^3(T)$$

and set

$$A = A(T) = 4/3 \, \pi \, \bar{A}(T) = \left(\frac{4\pi}{3}\right)^2 C \, P^3(T)$$

$$= \left(\frac{4\pi}{3}\right)^2 C \, P^3$$

From (5.4) it then follows that

$$(5.5) \qquad u^2(t) < u_0^2 + 6M^{2/3} \left(\frac{4\pi}{3}\right)^{2/3} C^{1/3} \, P$$

This inequality is satisfied for any solution to (3.3). In particular it is satisfied for $p(0) \epsilon \Omega(0)$, the support of the initial data G. Since

$$P = \sup_{p(0)\epsilon\Omega(0)} u(t), \quad t \epsilon [0,T]$$

it follows from (5.5) that an inequality for P is

$$P^2 < P_0^2 + 6M^{2/3} \left(\frac{4\pi}{3}\right)^{2/3} C^{1/3} P$$

$$P_0 = \sup_{p\epsilon\Omega(0)} E(p)$$

Thus an inequality for P is of the form

$$P^2 < P_0^2 + \beta M^{2/3}C^{1/3}P$$

where M and C are constants known a priori from the initial and background distributions. A bound for P is

$$P = P(T) < P_0 + \beta^2 M^{2/3}C^{1/3}$$

The bound is independent of T and is therefore uniform for all T. The uniform bound on support in velocity space leads to a bound on support in position space. From the general theorem we thus get the global-in-time existence and uniqueness of the classical solution to (1.1a,b) with data of the form (3.1).

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# Chapter 2: The Vlasov-Maxwell system

## I. Introduction

The purpose of the present paper is to give a local existence and uniqueness theorem for the Vlasov-Maxwell system of equations. The Vlasov-Maxwell system is one of the fundamental equations underlying the kinetic theory of plasma. It is a non-linear integro-differential system which describes the motion of a collection of charged particles in the presence of its own self generated electro-magnetic field. Under discussion is the following initial value problem

$$\text{a)} \quad \frac{\partial f}{\partial t} + v\cdot\nabla_x f - (E + v \times B)\cdot\nabla_v f = 0\ , \quad f(0) = g$$

$$(1.1) \qquad \text{b)} \quad \frac{\partial E}{\partial t} - \nabla \times B = 4\pi \int vf dv\ , \quad \frac{\partial B}{\partial t} + \nabla \times E = 0$$

$$E(0) = E_o ,\ B(0) = B_o$$

where

$$\text{c)} \quad \nabla\cdot E_o = -\ 4\pi \int g dv\ , \quad \nabla\cdot B_o = 0$$

$$E = (E_1,E_2,E_3)\ , \quad B = (B_1,B_2,B_3)$$

$$x = (x_1,x_2,x_3)\ , \quad v = (v_1,v_2,v_3)$$

The function, f, gives the number density of particles in phase space. The functions E, B are the electric and magnetic fields produced by the distribution, f. The system is internally consistent in that the distribution, f, evolves in time under the influence of the fields E, B, which are in turn being generated by f. Given initial condition (1.1c) it is easy to see that the solution (1.1) satisfies the additional equations

$$\nabla \cdot E = - 4\pi \int f dv \ , \ \nabla \cdot B = 0 \text{ for } t > 0 \ .$$

In the present context f represents a single species of charged particles, say electrons. Normally, a plasma is comprised of multiple species each satisfying a transport equation of the type (1.1a) and each contributing to the charge and current terms on the right side of equation (1.1b,c). The analysis given in this paper can be modified to deal with such an expanded multiple species system of equations. For the sake of simplicity we deal, however, with the single species system given by (1.1a,b,c).

The existence and uniqueness theorem for (1.1) is produced by generalizing a theorem of Kato [3, theorem II, p. 195]. In [3] Kato proves a local existence and uniqueness theorem for quasi-linear symmetric hyperbolic systems of equations. In the proof the author actually generalizes somewhat the type of systems being considered. The coefficients in the equations are not merely point wise operators on functions of x and t but are instead operators which are point wise in t but which are non-local in $x \in R_n$ mapping functions of $R_n$ into functions on $R_n$. In the present paper we further generalize the result in [3] so that the coefficients are non-local in both x and t mapping functions of $R_n x[0,T]$ into functions on $R_n x[0,T]$. The starting point of this analysis is the linear theorem [3, Theorem I, p. 189] for linear symmetric hyperbolic systems. This theorem is used as in [3] to prove local existence and uniqueness for a general system with operator coefficients of the type mentioned above. Solutions are obtained as continuous mappings of t into classes of Sobolev spaces on $R_n$. The proof given is in fact a modification of that in [3]. Having generalized Kato's work we can then give a proof for local existence and uniqueness for the Vlasov-Maxwell system. The proof we give is for an initial distribution with compact support. For this class of data the Vlasov-Maxwell system can be put into a form so that the general theorem applies.

As far as we know the existence and uniqueness theorem presented in this paper is the first such result obtained for the fully non-linear three-dimensional Vlasov-Maxwell system. Previously some results have been obtained for the system in lower dimensions and with various additional modifications or restrictions, [1], [2], [4].

## II. Function spaces and notation

We are adopting as much as possible the notation of [3]. Some chances in notation, however, are made.

$R_m$ - m dimensional Euclidean space

P - A real or complex Hilbert space which in the present context will be identified with $R_m$ for some value of m.

For $x \in R_m$, $|x| = (\sum_{i=1}^{m} x_i^2)^{1/2}$

$u(R_m)$ - a real P valued functions defined on $R_m$

$L^2(R_m,P)$ - the linear space of P valued functions on $R_m$ such that

$$||u||_{o,m} = ( \int |u|^2 dx)^{1/2} < \infty$$

$H^s(R_m,P)$ - the space of P value functions on $R_m$ for which the distributions derivatives of order $\leq$ s are of class $L^2(R_m,P)$.

The norm is

$$||u||_{s,m} = (\sum_{|\alpha| \leq s} C_\alpha ||D^\alpha u||^2_{o,m})^{1/2}$$

$C_\alpha$ - positive constants

$L^2_\ell(R_m,P)$ - the set of P valued functions on $R_m$ which are uniformly locally $L^2$ in the sense that

$$||u||_{o\ell,m} = \sup_{x\in R_m} \left(\int_{|x-y|<1} |u(y)|^2 dy\right)^{1/2} < \infty$$

$H^s_\ell(R_m,P)$ - the set of P valued functions on $R_m$ whose distribution derivatives of order $\leq$ s are of class $L^2_\ell(R_m,P)$. The norm is

$$||u||_{s\ell,m} = \sup_{|\alpha|\leq s} ||D^\alpha u||_{o\ell,m} \cdot$$

Let B(P) be the set of bounded linear operators on P. We are only interested in the case where P is finite dimensional in which case elements of B(P) are p×p matrices (p—dimension of P). For $a = [a_{i,j}] \in B(P)$ a suitable operator norm is

$$|a| = \left(\sum_{i,j} |a_{i,j}|^2\right)^{1/2}$$

$L^2_\ell(R_m,B(P))$ - the set of functions of $R_m$ valued in B(P) for which

$$||a||_{0\ell,m} = \sup_{x\in R_m} \left(\int_{|x-y|<1} |a(y)|^2 dy\right)^{1/2} < \infty$$

$H^s_\ell(R_m,B(P))$ - the set of functions on $R_m$ valued in $B(P)$ whose derivatives of order $\leq s$ are elements of $L^2_\ell(R_m,B(P))$.

$$||a||_{s\ell,m} = \left(\sum_{|\alpha|\leq s} C_\alpha ||D^\alpha a||^2_{0\ell,m}\right)^{1/2}$$

$C_\alpha$ - postive constants

$L^1[0,T;H^s(R_m,P)]$ - the set of $P$ valued integrable functions $u(t,\cdot)$ on $[0,T]\times R_m$ such that for each $t$ $u(t,\cdot) \in H^s(R_m,P)$.

$C^m[0,T;H^s(R_m,P)]$ - the set of $P$ valued function, $u(t,\cdot)$, on $[0,T]\times R_m$ such that the functions $D^\alpha u(t,\cdot)$, $|\alpha| \leq m$, are continuous mappings of $t$ into $H^s(R_m,P)$.

Similar definitions apply for mappings of $t$ into $H^s_\ell(R_m,P)$, $H^s_\ell(R_m,B(P))$. As normed the spaces $H^s(R_m,P)$ are Hilbert spaces $H^s_\ell(R_m,P)$, $H^s_\ell(R_m,B(P))$ are Banach spaces.

## III. Linear theorem

In this section we state the linear theorem on which the proofs for the non-linear results depend, this theorem is proved in [3]. The proof is rather complex and is an application of the theory of semigroups to evolution equations. We will not give the proof here but will simply state the result. For simplicity a somewhat less general theorem is stated but one which is adequate for our purposes.

The following Cauchy problem is considered

$$(3.1) \qquad \frac{\partial u}{\partial t} + \sum_{j=1}^{m} a_j(t,x) \frac{\partial u}{\partial x_j} = f(t,x)$$

$$0 \leq t \leq T \ , \ x \in R_m \ , \ u(0,x) = u_o(x)$$

the functions $u$ , $f$ are valued in the Hilbert space $P$ which in the present context is taken to be $R_n$ for some $n$ , the coefficients $a_j$ are in $B(P)$.

Theorem 3.1. Let $s$ be an integer such that $s > m/2 + 1$ , and let $1 \leq s' \leq s$ . Assume the following conditions:

$$(3.2) \qquad a_j \in C[0,T \ ; \ H_\ell^0 \ (R_m \ , \ B(P))]$$

$$(3.3) \qquad ||a_j(t)||_{s\ell,m} \leq K \quad 0 \leq t \leq T$$

(3.4) $a_j(t,x)$ is symmetric
for $0 \leq t \leq T$ , $x \in R_m$

$$(3.5) \qquad f \in L^1[0,T\ ;\ H^{s'}(R_m,P)] \cap C[0,T\ ;\ H^{s'-1}(R_m,P)]$$

$$(3.6) \qquad u_o \in H^{s'}(R_m,P)$$

then (3.1) has a solution u such that

$$(3.7) \qquad u \in C[0,T\ ;\ H^{s'}(R_m,P)] \cap C^1[0,T\ ;\ H^{s'-1}(R_m,P)]$$

the solution is unique in the larger class

$$C[0,T\ ;\ H^1(R_m,P)] \cap C^1[0,T\ ;\ H^0(R_m,P)]$$

the following estimates hold:

$$(3.8) \qquad ||u(t)||_{r,m} \leq e^{cKT} (||u_o||_{r,m} + cF_r)$$

$$(3.9) \qquad ||\frac{d}{dt} u(t)||_{r-1,m} \leq c(||f(t)||_{r-1,m} + K||u(t)||_{r,m}) ,$$

$$0 \leq t \leq T$$

where $0 \leq r \leq s'$ in (3.8) and $1 \leq r \leq s'$ in (3.9) and

$$(3.10) \qquad F_r = \int_0^T ||f(t)||_{r,m}\, dt$$

the constant c depends on m and s.

Proof. See [3, pp. 189 - 194].

## IV. General non-linear theorem

The linear theorem is now used to prove a non-linear existence and uniqueness theorem which is a modification of [3, Theorem II, p. 195]. Instead of the system $Q'$ [3, p. 195] we consider the system

$$(4.1) \qquad \frac{\partial u}{\partial t} + \sum_{j=1}^{m} G_j[u(t)] \frac{\partial u}{\partial x_j} = 0$$

$$u(0) = u_o$$

(For present purposes the operator $G_o$ multiplying $\frac{\partial u}{\partial t}$ can be set to one and the right hand side F to zero). The quantities $G_j(\cdot)$ are non-linear operators that carry functions valued in $L^1[0,T\ ;\ H^s(R_m,P)]$ into functions valued in $L^1[0,T\ ;\ H^s(R_m,B(P))]$. The difference here from $Q'$ is that in $Q'$ the operators $G_j$ act pointwise in t (with t as a parameter) mapping functions on $R_m$ into functions on $R_m$. In (4.1) the operators are non-local in both x and t and act on functions defined on $[0,T] x R_m$. With this generalization we can apply the theorem to the Vlasov-Maxwell system of plasma physics.

Theorem 4.1. Suppose $s > m/2 + 1$. Let D be a bounded open subset of $H^s(R_m,P)$, T an arbitrary positive number, and A(T) the set $L^1[0,T\ ;\ D] \cap C[0,T\ ;\ H^{s-1}(R_m,P)]$. Let $G_j(\cdot)$ be non-linear operators on A(T) to $L^1[0,T\ ;\ H^s_\ell(R_m,B(P))]$. Assume the following conditions hold:

(4.2) The quantities $G_j[h]$ are bounded (by a constant $\lambda$) in the

$H^s_\ell$ - norm for $h \in A(T)$, uniformly in $t \in [0,T]$ .

(4.3) For each j and $t \in [0,T]$ the map $h(t) \to G_j[h(t)]$ satisfies the condition

$$\sup_{r \le t} ||G_j[h(r)] - G_j[\bar{h}(r)]||_{o\ell,m}$$

$$\le \mu(T) \sup_{r \le t} ||h(r) - \bar{h}(r)||_{o,m}$$

$\mu(T)$ - constant depending on T.

(4.4) The maps $t \to G_j[h(t)]$ are continuous in the $H^o_\ell$-norm for each $h \in A(T)$ .

(4.5) For each $h \in A(T)$, $G_j[h](x) \in B(P)$ is symmetric for each $t, x \in [0,T] x R_m$ , $j = 1,.., m$

(4.6) $u_o \in D$

Then there is a unique solution of (4.1) defined on $[0,T']$ , where $0 < T' \le T$ such that

(4.7) $u \in C[0,T' ; D] \cap C^1[0,T'; H^{s-1}(R_m,P)]$

Proof. We are guided as much as possible by the proof of that for [3, Theorem II, p. 195]. Let us assume for simplicity that D is the set

$$\left\{ u/u \in H^{s}(R_m,P)\ ,\ ||u||_{s,m} < R \right\}$$

and that $||u_o||_{s,m} < R/2$. The argument for more general open sets D can be done as in [3]. Let S be the set of functions, v, on $[0,T']$, $T' \leq T$, to $H^{s}(R_m,P)$ such that

$$||v(t)||_{s,m} \leq R$$

$$v(t) \subset C[0,T' ; H^{s-1}(R_m,P)]$$

The constant $T'$ is to be determined later. With the metric

$$|||v - \bar{v}||| = \sup_{0\leq t\leq T'} ||v(t) - \bar{v}(t)||_{o,m}$$

S is a complete metric space (see [3, Lemmas 2.15, 2.18, p. 188]).

If $v \in S$ then $v \in A(T')$ and we can form the functions

$$g_j^{v} = G_j[v(t)] \qquad (4.8)$$

If we consider the linear system (4.1L) formed by replacing the coefficients in (4.1) by (4.8) we see immediately that the conditions of Theorem (3.1) are satisfied, i.e. conditions (3.2), (3.3), (3.4) are satisfied by virtue of (4.4), (4.2), (4.5). Let w be the solution to (4.1,L). It follows by (3.8), (3.9) that w has the bounds

$$||w||_{s,m} \leq e^{c\lambda T'} \; ||u_o||_{s,m}$$

$$||\frac{d}{dt} w||_{s-1,m} \leq c\lambda e^{c\lambda T'} \; ||u_o||_{s,m}$$

For $T' \leq \ln(2)/c\lambda$ then $w \in S$ .

We define the map $\phi(v) = w$ from S into itself and show that if $T'$ is sufficiently small $\phi$ is a contraction. Let

$$w = \phi(v)$$

$$\bar{w} = \phi(\bar{v}) \quad , \quad v \, , \, \bar{v} \in S$$

Then $(w - \bar{w})$ satisfies the system

$$\frac{\partial}{\partial t} (w - \bar{w}) + \sum_{j=1}^{m} G_j[v] \frac{\partial}{\partial x_j} (w - \bar{w})$$

$$= - \sum_{j=1}^{m} (G_j[v] - G_j[\bar{v}]) \frac{\partial \bar{w}}{\partial x_j}$$

$$(w - \bar{w})(0) = 0$$

Since $v \in S$ then $||G_j[v]||_{s\ell,m} \leq \lambda$ . Since $\bar{w} \in S$ then $\bar{w}_{x_j} \in H^{s-1}$ , $s-1 > m/2$ ,

$$||\bar{w}_{x_j}||_{s-1,m} \leq R \, .$$

It follows from Theorem (3.1) and [3, Lemma 2.11, pp. 185 - 186] that for $0 \le t \le T' \le T$

$$||(w - \bar{w})(t)||_{o,m} \le ce^{c\lambda t} \int_{o}^{t} \bar{c} \sum_{j=1}^{m} ||G_j[v] - G_j[\bar{v}]||_{o,\ell m} ||\bar{w}_{x_j}||_{s-1} d\bar{t}$$

$$\le c\bar{c}\ e^{c\lambda T'} R \int_{o}^{T'} \sum_{j=1}^{m} ||G_j[v] - G_j[\bar{v}]||_{o\ell,m}\ d\bar{t}$$

$$\le c\bar{c}e^{c\lambda T'} mR\mu(T) \int_{o}^{T'} \sup_{s\le\bar{t}} ||v(s) - \bar{v}(s)||_{o,m}\ d\bar{t}$$

$$\le c\bar{c}e^{c\lambda T'} mR\mu(T)T' \sup_{s\le T'} ||v(s) - \bar{v}(s)||_{o,m}$$

$$\le c\bar{c}e^{c\ T'} mR\mu(T)T' |||v - \bar{v}|||$$

$c,\bar{c}$ - Constants

Taking the supremum for $t \le T'$ of the left side of the above inequality gives

$$|||w - \bar{w}||| \le c\bar{c}e^{c\lambda T'} mR\mu(T)T' |||v - \bar{v}|||$$

For $T'$ sufficiently small

$$c\bar{c}e^{c\lambda T'} mR\mu(T)T' < \alpha < 1$$

Thus

$$|||w - \bar{w}||| \leq \alpha \, |||v - \bar{v}|||$$

and $\phi$ is a contraction. Thus $\phi$ has a unique fixed point u in S which is then a solution to (4.1). That u is the unique solution satisfying (4.7) follows from Theorem (3.1) and the fact that any solution satisfying (4.7) is a fixed point of $\phi$ .

## V. The Vlasov-Maxwell system

In this section Theorem 4.1 is used to prove local existence and uniqueness of solutions to the Vlasov-Maxwell system (1.1). The assumption will be made that the function g has compact support. By taking account of the boundedness of the support of the solution it is possible to put this problem into a form so that Theorem 4.1 applies. The proof gives solutions in a class of Sobolev spaces with sufficiently high order of $L^2$ differentiability so that by Sobolev's lemma strict solutions to the problem are obtained.

In this section functions are defined on $R_6$ or $R_3$ and are valued in $R_1, R_3$, or $R_6$. Where it is obvious we suppress reference to the dimension of the range space of functions in the notation, i.e., $H^s(R_6,R_1) = H^s(R_6)$, $H^s(R_3,R_3) = H^s(R_3)$. Let us assume for the initial data to (1.1a,b) that

$$g \in H^s(R_6) \quad s \geq 5$$

$$E_o, B_o \in H^s(R_3) \quad s \geq 5$$

(5.1)

$$\text{supp } g \subset \{(x,v)/|x| \leq R,\ |v| \leq R\}$$

$$||g||_{s,6} \leq M$$

$$||E_o||_{s,3},\ ||B_o||_{s,3} \leq N$$

R,M,N - constants

Theorem 5.1. Let $s \geq 5$ and the initial conditions to (1.1a,b) satisfy (5.1). There is a positive number $T_o$ and a unique function

$$f \in C[0,T_o;H^s(R_6)] \cap C^1[0,T_o;H^{s-1}(R_6)]$$

such that f is a solution to (1.1a,b).

Proof. Let the function $\alpha(v)$ be defined as follows

$$\alpha(v) = \left(\alpha_1(v)\ ,\ \alpha_2(v)\ ,\ \alpha_3(v)\right) \text{ where}$$

$$\alpha_i(v) = \begin{array}{l} v_i\ ,\ |v| \leq 2R \\ \\ 0\ ,\ |v| \geq 3R \end{array}$$

$\alpha_i(v) \in C_o^\infty(R_3)$ (infinitely differentiable function with compact support).

The system 1.1 is replaced with the following system

$$\text{a)}\quad \frac{\partial f}{\partial t} + \alpha(v)\cdot\nabla_x f - (E + \alpha(v)\times B)\cdot\nabla_v f = 0$$

$$f(0) = g$$

(5.2)

$$\text{b)}\quad \frac{\partial E}{\partial t} - \nabla\times B = 4\pi \int_{|v|\leq 2R} vf\,dv$$

$$\frac{\partial B}{\partial t} + \nabla\times E = 0$$

$$E(0) = E_o,\ B(0) = B_o$$

$E_o, B_o$ satisfy 1.1c

Let D,A be the following sets of functions

$$D = \{h/h \in H^s(R_6,R_1)\ ,\ ||h||_{s,6} < 2M\}$$

$$A = L^1[0,T;D] \cap C[0,T;H^{s-1}(R_6,R_1)]$$

Assume $s \geq 5$ and T an arbitrary positive number.

For $h \in A$ we solve Maxwell's equation (5.2b) with the integral involving f on the right hand side replaced by

$$4\pi \int_{|v| \leq 2R} vhdv \; .$$

Clearly the functions

$$H_i[h](x,t) = 4\pi \int_{|v| \leq 2R} v_i hdv \quad \text{are}$$

of class $L^1[0,T;H^s(R_3)] \cap C[0,T;H^{s-1}(R_3)]$ , $s \geq 5$ , where for each $t \in [0,T]$

$$(5.3) \qquad ||H_i[h]||_{s,3} \leq C(R) \; ||h||_{s,6} \leq C(R)(2M)$$

C(R) - a constant depending on R.

In order to apply Theorem (3.1) we write Maxwell's equations in the form of a symmetric hyperbolic system.

Let

$$\bar{Y} = [E,B]^T \; , \; \bar{Y}_o = [E_o,B_o]^T$$

$$\bar{H}[h] = [H_1,H_2,H_3,0,0,0]^T$$

In terms of the vector $\bar{Y}$ Maxwell's equations (5.2b) with right hand side $\bar{H}[h]$ takes the form

$$(5.4) \qquad \frac{\partial}{\partial t} \bar{Y} + \sum_{i=1}^{3} A_i \frac{\partial \bar{Y}}{\partial x_i} = \bar{H}[h]$$

$$\bar{Y}(0) = \bar{Y}_o$$

where $A_i$ are symmetric constant matrices. For $h \in A$

$$\bar{H}[h] \in L^1[0,T;H^s(R_3,R_6)] \cap C[0,T;H^{s-1}(R_3,R_6)]$$

where from (5.3)

$$||\bar{H}[h]||_{s,3} \leq 6C(R)(2M)$$

The vector $\bar{Y}_o \in H^s(R_3,R_6)$ , $||\bar{Y}_o||_{s,3} \leq 2N$ . The constant matrices $A_i$ satisfy

$$||A_i||_{s\ell,3} \leq K = 4(\pi/3)^{1/2} \ , \ s \geq 5 \ .$$

From Theorem (3.1) we have that the solution to (5.4), call it $\bar{Y}[h]$ , exists and is unique as an element of

$$C[0,T;H^s(R_3)] \cap C^1[0,T;H^{s-1}(R_3)]$$

and that

$$||\bar{Y}[h]||_{s,3}(t) \le e^{cKt}(||\bar{Y}_o||_{s,3} + c\int_o^t ||\bar{H}[h]||_{s,3}(\bar{t})\,d\bar{t})$$

$$0 \le t \le T$$

It follows that

$$(5.5) \qquad ||\bar{Y}[h]||_{s,3}(t) \le e^{cKt}(2N + ct6(C(R))(2M))$$

$$\le e^{cKT}(2N + cT6(C(R))(2M))$$

and from (5.4) that

$$(5.6) \qquad ||\bar{Y}[h] - \bar{Y}[\bar{h}]||_{o,3}(t)$$

$$\le ce^{cKt}\int_o^t ||\bar{H}[h] - \bar{H}[\bar{h}]||_{o,3}(\bar{t})\,d\bar{t}$$

$$\le 6ce^{cKt}C(R)\int_o^t ||h - \bar{h}||_{o,6}(\bar{t})\,d\bar{t}$$

$$t \in [0,T]\ ,\ h\ ,\ \bar{h} \in A$$

Let $E[h](x,t)$ , $B[h](x,t)$ be the fields obtained by solving Maxwell's equations with the right hand side for formed with $h$ . We form the following non-linear operators carrying functions in A into functions in

$$L^1[0,T;H_\ell^s(R_6,B(R_1))]$$

(elements of $B(R_1)$ are simply multiplies)

$$G_i[h](x,v,t) = \alpha_i(v)$$

$$G_{i+3}[h](x,v,t) = (E[h] + \alpha(v) \times B[h])_i \ , \ i = 1,2,3$$

One now easily shows that the operators $G_1$ , --, $G_6$ satisfy conditions (4.2), --, (4.5). For $G_i$ , $i = 1,2,3$ this is obvious since $\alpha_i \in C_o^\infty$ and is independent of $h$ . For $i = 4$ , - , 6 (4.2) is satisfied by virtue of (5.5) taking into account the bound for $\alpha_i(v)$ . Condition (4.4) follows from the fact that

$$\bar{Y}[h] = [E[h] \ , \ B[h]]^T \in C[0,T;H^s(R_3)]$$

Condition (4.5) is obvious since $G_i$ are scalars. Condition (4.3) follows from (5.6) where

$$\mu(T) = 6c\bar{c} \ e^{cKT}C(R)$$

$\bar{c}$ - a constant depending on $\alpha(v)$ .

It follows from Theorem 4.1 that the system (5.2) has a unique solution f of class

$$C[0,T';D] \cap C^1[0,T';H^{s-1}(R_6)]$$

for some $0 < T' \leq T$ .

Let E[f] , B[f] be the field associated with the solution f to system (5.2). From Sobolev's lemma (as stated for example in [3, Lemma 2.1, p. 182] one ascertains that $E_i[f]$ , $B_i[f]$ are of class $C^1(R_3 \times [0,T']$ and are uniformly bounded by a constant $\lambda'$ on the interval $[0,T']$ . Using bounds on E and B and the fact that the initial data has support in the set

$$\{(x,v) \ , \ |x| \leq R \ , \ |v| \leq R\} \ ,$$

one integrates the characteristic equations for (5.2) to obtain bounds on the support of the solution. The characteristic equations for (5.2) are

$$\frac{d\bar{x}}{dt} = \bar{v}$$

$$(5.7) \qquad \frac{d\bar{v}}{dt} = (\bar{E} + \tilde{B}\bar{v})$$

(as long as $|v(t)| \leq 2R$ along the trajectory). Here

$$\bar{x} = (x_1, x_2, x_3)^T$$

$$\bar{v} = (v_1, v_2, v_3)^T$$

$$\bar{E} = (E_1, E_2, E_3)^T$$

$$\tilde{B} = \begin{bmatrix} 0 & B_3 & -B_2 \\ B_3 & 0 & -B_1 \\ B_2 & -B_1 & 0 \end{bmatrix}$$

For $(x,v) \in R_6$ let $P(x,v) = |v|$ . Let $\Omega(t) = \sup_{R_6} f(t)$ - the support of f in $R_6$ at time t. We define the function

$$q(t) = \sup_{\Omega(t)} P(x,v)$$

which bounds the support of f in velocity space at time t. Integrating (5.7) for initial points in the support of the initial data g we have that

$$(5.8) \qquad |v(t)| \leq |v(0)| + \int_0^t |E(\bar{t})| + |\tilde{B}(\bar{t})||v(\bar{t})| \, d\bar{t}$$

$$\leq R + 6\lambda' \int_0^t (1 + q(\bar{t}))\, d\bar{t}$$

Taking the supremum over all such trajectories originating in the support of g gives the integral inequality for q(t)

$$(5.9) \qquad q(t) \le R + 6\lambda' \int_0^t (1 + q(\bar{t}) \, d\bar{t}$$

$$= R + 6\lambda' t + 6\lambda' \int_0^t q(\bar{t}) \, d\bar{t}$$

the solution is

$$q(t) \le Re^{6\lambda' t} + (e^{6\lambda' t} - 1)$$

This gives a bound on the support in velocity space of the solution f valid for an interval of time $[0,T_o]$ , $T_o \le T'$ for which $q(t) \le 2R$ .

The time $T_o$ is obtained by solving the inequality

$$Re^{6\lambda' t} + (e^{6\lambda' t} - 1) \le 2R$$

the solution is

$$t \le 1/6\lambda' \ \ln ((2R + 1)/(R + 1)) = T''$$

setting

$$T_o = \min (T' , T'')$$

we have that for $t \in [0,T_o]$

$$\text{supp } f(t) \subset \{(x,v)/|v| \leq 2R\}$$

Thus for $t \in [0,T_o]$ the solution to (5.2) coincides with the solution to (1.1). It follows that for the interval $[0,T_o]$ the function f is the unique solution of class

$$C[0,T_o;D] \cap C^1[0,T_o;H^{s-1}(R_6)] \text{ , } s \geq 5 \text{ ,}$$

to system (1.1).

It is an application of Sobolev's lemma that the solution obtained in Theorem 5.1 is a strict solution to the problem.

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